Linear absolute value relation analysis

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Overview

- Motivation
- Double description method for linear absolute value systems
- An abstract domain of linear absolute value inequalities
- Implementation and Experiments
- Conclusion

Motivation

Numerical static analysis by abstract interpretation

Numerical static analysis

- discover numerical properties of a program statically and automatically

Theoretical framework: **abstract interpretation**

to design static analyses that are

- sound by construction (no behavior is omitted)
- approximate (trade-off between precision and efficiency)

Numerical abstract domains

- infer relationships among numerical variables
- examples
  - Intervals ($a \leq x \leq b$), Octagons ($\pm x \pm y \leq c$), Polyhedra ($\Sigma_k a_k x_k \leq b$)

Polyhedra and Sub-polyhedra abstract domains

The polyhedra abstract domain [Cousot Halbwachs 78]

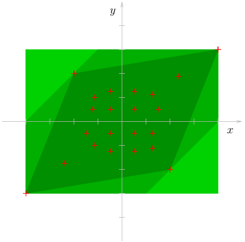
- linear relation analysis to infer linear invariants

$$\bigwedge \Sigma_i a_i x_i \leq b$$

  where $a_i, b \in \mathbb{I}$ and $\mathbb{I} \in \{\mathbb{Q}, \mathbb{R}\}$
- implementations
  - Polylib, NewPolka (in APRON), PPL, $\cdots$

Sub-polyhedra abstract domains

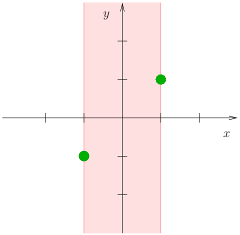
- octagons ($\pm x \pm y \leq c$) [Miné 01]
- octahedra ($\Sigma_i \pm x_i \leq c$) [Clarisó et al. 04]
- TVPI ($ax_i + bx_j \leq c$) [Simon et al. 03]
- template polyhedra [Sankaranarayanan et al. 05] ($\Sigma_i a_i x_i \leq c$ where a_i are fixed beforehand)
- ...



Motivation

Convexity limitations: a motivating example

```
real x, y;
x ← 1;
y ← 1;
while (true) {
    x ← −x;
    y ← 1/x;   ①
}
```

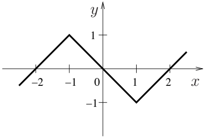


Loc	Most abstract domains	Concrete semantics
①	$x \in [-1, 1]$ $y \in [-\infty, +\infty]$	$(x = -1 \wedge y = -1)$ $\vee(x = 1 \wedge y = 1)$

Division-by-zero? Safe !

Motivation

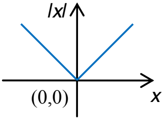
Piecewise linear $\longrightarrow$ Non-linear $\updownarrow$ Linear



Absolute Value (AV): $y = |x|$

- piecewise linear expressiveness

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Possible applications

- AV functions in C: $abs(), fabs(), \ldots$
- MiniMax functions in C: $fmax(), fmin(), \ldots$
  - e.g., $\max(x, y) = \frac{1}{2}(|x - y| + x + y)$
- Abstractions for floating-point rounding errors
  - $|R_{\mathbf{f},r}(x) - x| \leq \varepsilon_{\mathrm{rel}} \cdot |x| + \varepsilon_{\mathrm{abs}}$ (float: $\varepsilon_{\mathrm{rel}} = 2^{-23}, \varepsilon_{\mathrm{abs}} = 2^{-149}$)

Double Description Method for AVI systems

Equivalence among itv linear, linear AVI, XLCP systems

3 kinds of equivalent relations:

- interval linear inequalities (ILI): $\sum_k [a_k, b_k] x_k \le c$ [Chen et al. SAS'09]
- linear absolute value inequalities (AVI): $\sum_k a'_k x_k + \sum_k b'_k |x_k| \le c'$
- extended linear complementary problem (XLCP) inequalities:

$$\sum_k a''_k x_k^+ + \sum_k b''_k x_k^- \le c''$$

where x_k^+, x_k^- satisfy

$$x_k^+, x_k^- \ge 0 \text{ and } \sum_k x_k^+ x_k^- = 0.$$

  - $x_k^+ = 0 \vee x_k^- = 0$
  - $x_k = x_k^+ - x_k^-, |x_k| = x_k^+ + x_k^-;$
  - $x_k^+ = \frac{1}{2}(x_k + |x_k|), x_k^- = \frac{1}{2}(|x_k| - x_k);$

Example

AVI: $\{|x| \le 1, -|x| \le -1\}$, ILI: $\{x \le 1, -x \le 1, [-1,1]x \le -1\}$,
XLCP: $\{x^+ + x^- \le 1, -x^+ - x^- \le -1, x^+ \ge 0, x^- \ge 0, (x^+)^T x^- = 0\}$

Double Description Method for Polyhedra

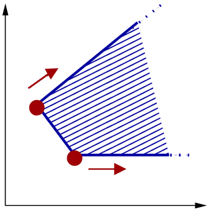
Theorem (Minkowski-Weyl Theorem)

The set $P \subseteq \mathbb{R}^n$ is a polyhedron, iff it is finitely generated, i.e., there exist finite sets $V, R \in \mathbb{R}^n$ such that P can be generated by (V, R):

$$P = \left\{ \sum_{i=1}^{|V|} \lambda_i V_i + \sum_{j=1}^{|R|} \mu_j R_j \;\middle|\; \forall i, \lambda_i \geq 0, \forall j, \mu_j \geq 0, \sum_{i=1}^{|V|} \lambda_i = 1 \right\}$$

Dual representations

- constraint representation: $Ax \leq b$
  - e.g., $\{-y \leq -1, x - y \leq 1, -x - y \leq -3\}$
- generator representation: $G = (V, R)$
  - e.g., $(\{(2,1),(1,2)\}, \quad \{(0,1),(1,1)\})$



Dual conversion: Chernikova's algorithm

XLCP: From Constraints to Generators

XLCP: $Mx^+ + Nx^- \leq c \wedge x^+, x^- \geq 0 \wedge (x^+)^T x^- = 0$
Step1: $G \leftarrow$ Polyhedra.Cons2Gens $(Mx^+ + Nx^- \leq c \wedge x^+, x^- \geq 0)$
Step2: $G^c \leftarrow \{g \in G \mid g \text{ satisfies } (x_g^+)^T x_g^- = 0\}$
Step3: $G^{cc} \leftarrow \{< G^c_{s_1}, \ldots, G^c_{s_i}, \ldots, G^c_{s_m} >\}$ where $G^c_{s_i} = (V^c_{s_i}, R^c_{s_i})$ satisfies

1. $V^c_{s_i} \subseteq V^c$, $R^c_{s_i} \subseteq R^c$, $\cup_{i=1}^m V^c_{s_i} = V^c$, $\cup_{i=1}^m R^c_{s_i} = R^c$, and
2. Within each group $G^c_{s_i}$, any sum z of an arbitrary convex combination of extreme points from $V^c_{s_i}$ and an arbitrary nonnegative combination of extreme rays from $R^c_{s_i}$, satisfies the complementary condition $(z^+)^T z^- = 0$.

Theorem

Let $P_\pm = \{x \in \mathbb{R}^{2n} \mid Ax \geq b, x \geq 0, (x^+)^T x^- = 0\}$, and let $G^{cc} = \langle G^c_{s_1}, \ldots, G^c_{s_i}, \ldots, G^c_{s_m} \rangle$ be the grouping result of its complementary generators where $G^c_{s_i} = (V^c_{s_i}, R^c_{s_i})$. Then $x \in P_\pm$, iff there exists some i $(i \in \mathbb{N}, 1 \leq i \leq m)$ such that

$$x = \sum_{v_j^c \in V^c_{s_i}} \lambda_j v_j^c + \sum_{r_k^c \in R^c_{s_i}} \mu_k r_k^c$$

where $\lambda_j, \mu_k \geq 0, \Sigma_j \lambda_j = 1$.

XLCP: From Constraints to Generators (cont.)

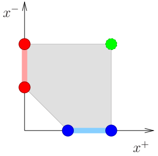
Example

XLCP: $\{-x^+ - x^- \leq -1, x^+ \leq 2, x^- \leq 2, x^+ \geq 0, x^- \geq 0, (x^+)^T x^- = 0\}$
Polyhedral generators of $\{-x^+ - x^- \leq -1, x^+ \leq 2, x^- \leq 2, x^+ \geq 0, x^- \geq 0\}$:

$$(V, R) = \left(\begin{pmatrix} x^+ \\ x^- \end{pmatrix} : \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \emptyset \right)$$

Grouping results of complementary generators G^{cc}:

$$\left\{ \left(\begin{pmatrix} x^+ \\ x^- \end{pmatrix} : \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}, \emptyset \right), \left(\begin{pmatrix} x^+ \\ x^- \end{pmatrix} : \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}, \emptyset \right) \right\}$$



XLCP: From Constraints to Generators (cont.)

XLCP: $Mx^+ + Nx^- \leq c \wedge x^+, x^- \geq 0 \wedge (x^+)^T x^- = 0$

Step1: $G \leftarrow$ Polyhedra.Cons2Gens $(Mx^+ + Nx^- \leq c \wedge x^+, x^- \geq 0)$

Step2: $G^c \leftarrow \{g \in G \mid g \text{ satisfies } (x_g^+)^T x_g^- = 0\}$

Step3: $G^{cc} \leftarrow \{< G^c_{s_1}, \ldots, G^c_{s_i}, \ldots, G^c_{s_m} >\}$

Fortunately, when designing the AV abstract domain, we only need G^c!

XLCP: From Generators to Constraints

Step 1. $Mx^+ + Nx^- \leq b \leftarrow \text{Polyhedra.Gens2Cons}(\mathcal{G}^c)$;

Step 2. add $x^+, x^- \geq 0, (x^+)^T x^- = 0$

An abstract domain of linear absolute value inequalities

The AVI abstract domain

An abstract domain of linear absolute value inequalities (AVI)

- goal: to infer linear relations among values and absolute values of program variables

$$\Sigma_k a_k x_k + \Sigma_k b_k |x_k| \leq c$$

Domain representation for domain element **P**

- representation: a linear AVI system $Ax + B|x| \leq c$
- semantics: $\gamma(\mathbf{P}) = \{x \in \mathbb{R}^n \colon Ax + B|x| \leq c\}$

Topological properties: can be non-convex, even unconnected

- a (possibly empty) convex polyhedron in each orthant
- e.g., $-|x| \leq -1$

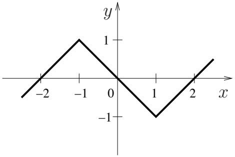
The AVI abstract domain (representation)

Expressiveness limitation: $\Sigma_k a_k x_k + \Sigma_k b_k |x_k| \leq c$

- $|\cdot|$ applies to only (single) variables rather than expressions

An example: $y = x - |x+1| + |x-1|$, i.e.,

$$y = \begin{cases} x+2 & \text{if } x \leq -1 \\ -x & \text{if } -1 \leq x \leq 1 \\ x-2 & \text{if } x \geq 1 \end{cases}$$



Expressiveness lifting

- introduce new auxiliary variables to denote expressions inside the AV function
- e.g., $\{y = x - |\nu_1| + |\nu_2|, \nu_1 = x+1, \nu_2 = x-1\}$

The AVI abstract domain (operations)

How to implement AVI domain operations for static analysis

- maintain the map between abstract environments over x and abstract environments over x^+, x^-:

$$x = x^+ - x^-, \qquad |x| = x^+ + x^-$$
$$x^+ = \frac{1}{2}(x + |x|), \qquad x^- = \frac{1}{2}(|x| - x)$$

where x^+, x^- satisfy $x^+ \geq 0, x^- \geq 0, (x^+)^T x^- = 0$

- let $G^c = (V^c, R^c)$ be the set of complementary generators of XLCP system:

$$Mx^+ + Nx^- \leq b$$
$$x^+ \geq 0, x^- \geq 0, (x^+)^T x^- = 0$$

The AVI abstract domain (operations)

Domain operations

1. lattice operations
   - emptiness test: **P** is empty, iff $V^c = \emptyset$
   - inclusion test: $\mathbf{P} \sqsubseteq \mathbf{P}'$ that is $\gamma(\mathbf{P}) \subseteq \gamma(\mathbf{P}')$, iff
     $\forall v \in V^c, M'\, v^+ + N'\, v^- \leq b' \quad \wedge \quad \forall r \in R^c, M'\, r^+ + N'\, r^- \leq 0$
   - meet: $\mathbf{P} \sqcap \mathbf{P}'$ is an AVI domain element whose XLCP system is
     $$\begin{aligned} Mx^+ + Nx^- &\leq b \\ M'x^+ + N'x^- &\leq b' \\ x^+ \geq 0, x^- \geq 0, (x^+)^T x^- &= 0 \end{aligned}$$
   - join: $\mathbf{P} \sqcup \mathbf{P}'$ is the least AVI domain element containing **P** and **P**′, whose set of complementary generators is the union of those of **P** and **P**′: $(V^c \cup V'^c, R^c \cup R'^c)$.

The AVI abstract domain (operations)

Domain operations

2. transfer functions
   - test transfer function: $\tau[\![cx + d|x| \leq e]\!]^\sharp(\mathbf{P})$, whose XLCP system is defined as

$$\begin{aligned} Mx^+ + Nx^- &\leq b \\ (c+d)x^+ + (d-c)x^- &\leq e \\ x^+ \geq 0, x^- \geq 0, (x^+)^T x^- &= 0 \end{aligned}$$

   - projection: $\tau[\![x_j := random()]\!]^\sharp(\mathbf{P})$, whose set of complementary generators is defined as $(V^c, R^c \cup \{e_j^+, e_j^-, -e_j^+, -e_j^-\})$, where $e_j^\pm$ denotes a canonical basis vector
   - assignment transfer function: $\tau[\![x_j := \Sigma_i a_i x_i + \Sigma_i b_i |x_i| + c]\!]^\sharp(\mathbf{P})$, can be implemented as:

$$\left(\tau[\![x_j := random()]\!]^\sharp \circ \tau[\![\Sigma_i a_i x_i + \Sigma_i b_i |x_i| + c - x_j' = 0]\!]^\sharp(\mathbf{P})\right)[x_j'/x_j]$$

The AVI abstract domain (operations)

Domain operations

3. widening: given two AVI domain elements $\mathbf{P} \sqsubseteq \mathbf{P}'$, we define

$$\mathbf{P} \triangledown \mathbf{P}' \stackrel{\text{def}}{=} \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{x^+, x^- \geq 0, (x^+)^T x^- = 0\}$$

where

$$\mathcal{S}_1 = \{\, \varphi_1 \in (Mx^+ + Nx^- \leq b) \mid \mathbf{P}' \models \varphi_1 \,\},$$
$$\mathcal{S}_2 = \left\{ \varphi_2 \in (M'x^+ + N'x^- \leq b') \;\middle|\; \begin{array}{l} \exists \varphi_1 \in (Mx^+ + Nx^- \leq b), \\ \gamma(\mathbf{P}) = \gamma((\mathbf{P} \setminus \{\, \varphi_1 \,\}) \cup \{\, \varphi_2 \,\}) \end{array} \right\}$$

Implementation and Experiments

Prototype

Prototype implementation rAVI using:

- GMP (the GNU Multiple Precision arithmetic library)
  - to guarantee the soundness of the implementation
- NewPolka: a rational implementation of the polyhedra domain
  - for Chernikova's algorithm

Interface:

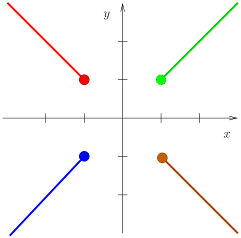
- plugged into the APRON library [Jeannet Miné]
- programs analyzed with INTERPROC [Jeannet et al.]

Comparison with

- NewPolka [Jeannet]
- itvPol: floating-point implementation of interval polyhedra [Chen et al. SAS09]

Example analyses

real x, y;
assume $x = 1$ or $x = -1$;
assume $y = 1$ or $y = -1$;
while $(true)$ {
① if $(x \geq 0)$ { $x := x + 1$; }
else { $x := x - 1$; }
if $(y \geq 0)$ { $y := y + 1$; }
else { $y := y - 1$; }
}



Loc	NewPolka	*itvPol*	rAVI
①	$\top$ (no information)	$[-1,1]x \leq -1 \wedge [-1,1]y \leq -1$	$\lvert x\rvert = \lvert y\rvert \wedge \lvert x\rvert \geq 1$

Preliminary experimental results

Program		NewPolka		itvPol		rAVI		Res. Inv.	
name	#vars	#iter.	$t(ms)$	#iter.	$t(ms)$	#iter.	$t(ms)$		
AVtest1	2	4	11	4	45	4	48	$<$	$<$
AVtest2	2	4	8	3	14	4	31	$<$	$<$
AVtest3	2	4	9	4	16	5	73	$<$	$<$
CmplxTest1	5	4	7	4	26	4	57	$<$	$<$
CmplxTest2	5	6	10	6	34	6	150	$<$	$<$
CmplxTest3	8	4	17	4	242	4	310	$<$	$<$
program4	1	5	2	4	4	4	10	$<$	$=$
program5	2	6	9	5	20	8	45	$<$	$<$

Most linear AV invariants captured by rAVI are essentially due to piecewise linear behaviors in the program, e.g., branches inside loops, case by case discussions over the difference between loop counter and input parameter (or initial value).

Conclusion

Summary:

- **goal**: handle **piecewise linear** behaviors in programs (non-convex)
- **approach**: linear absolute value relation analysis
  - show equivalence among itv linear, linear AV, extended LCP systems
  - develop a double description method for extended LCP
  - propose a new abstract domain: the AVI abstract domain

$$(\Sigma_k a_k x_k + \Sigma_k b_k |x_k| \leq c)$$

- can express non-convex (even unconnected) properties
- generalize the classical polyhedra abstract domain

Conclusion

Future Work

- **for precision**
  - automatic methods to introduce auxiliary variables on the fly that can be used inside the AV function
- **for efficiency**
  - weakly relational abstract domains over absolute value, with less expressiveness but higher efficiency
  - floating-point implementation
- **new applications**
  - program analysis of AV-related mathematical library functions
    - *abs*, *fdim*, *fmax*, *fmin*
  - piece-wise linear abstraction for floating-point arithmetic
  - analysis and verification of piece-wise linear (hybrid) systems